# Configurations of points and topology of real line arrangements

Juan Viu-Sos

**(joint work with Benoît Guerville-Ballé)**

Congreso bienal de la RSME 2017, Zaragoza

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PART I

# INTRODUCTION

## Line arrangements: geometry and combinatorics

What is a LINE ARRANGEMENT?

**Definition**

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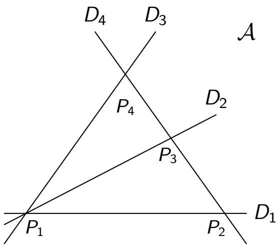

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$\mathcal{A}$ is *complexified real* if there exists a system of coordinates of $\mathbb{C}P^2$ such that any $D \in \mathcal{A}$ is defined by a $\mathbb{R}$-linear form.

## Why do we study LINE ARRANGEMENTS?

"Simple" case of reducible algebraic plane curves:

- $\mathcal{Q}_{\mathcal{A}} = \prod_{D\in\mathcal{A}} \alpha_D$, where $\alpha_D$ linear form such that $D = \alpha_D^{-1}(0)$.

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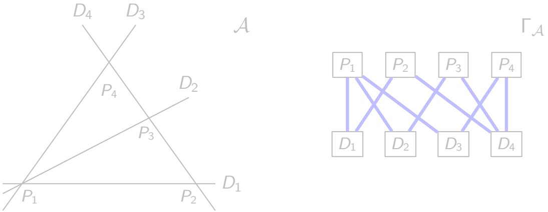

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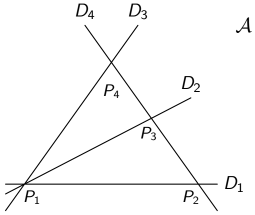


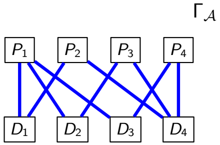

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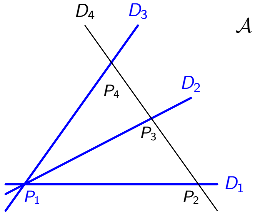


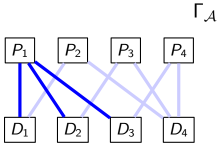

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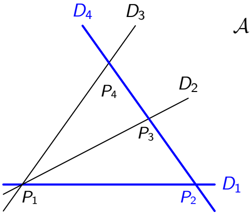


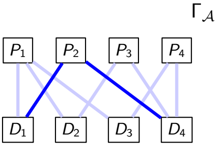

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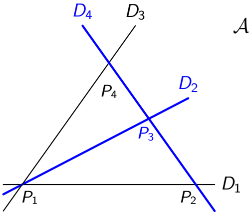


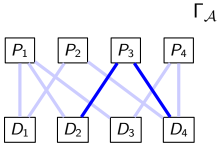

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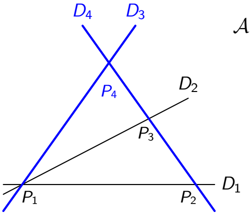


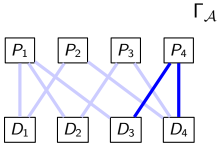

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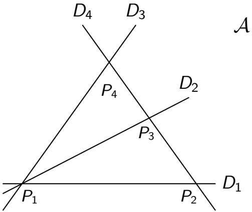


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Introduced by Artal, Florens and Guerville-Ballé in 2014.

### Definition

A *triangular inner-cyclic* arrangement $(\mathcal{A}, \gamma, \xi)$

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**Theorem (Artal-Florens-GB, GB-Meilhan)**

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Part II

# Configurations of points

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The tuple $\mathcal{C} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ is a $(t, m)$–configuration if:

1. $\forall V_i, V_j \in \mathcal{V} : \mathcal{S} \cap (V_i, V_j) = \emptyset$,
2. $\mathcal{V} = \mathrm{pl}^{-1}(0)$,
3. $\forall L \in \mathcal{L} : \sum_{S \in L} \mathrm{pl}(S) = 0$.

# Configurations of points

We take in $\mathbb{R}P^2$ :

- $\mathcal{V} = \{V_1, \ldots, V_t\}$ *vertices*,
- $\mathcal{S} = \{S_1, \ldots, S_n\}$ *surrounding-points*,
- $\mathcal{L} = \{L = (S, V) \mid S \in \mathcal{S},\ V \in \mathcal{V}\}$ collection of lines,
- a weight assignment $\mathrm{pl} : \mathcal{V} \sqcup \mathcal{S} \to \mathbb{Z}_m$.

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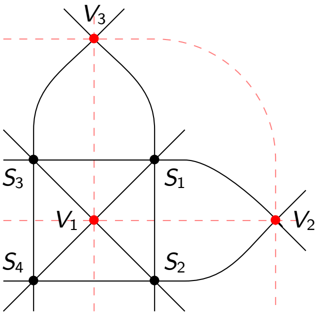
A $(3, 2)$-configuration :




$$\text{pl} : (S_1, S_2, S_3, S_4) \mapsto (1, 1, 1, 1) \in \mathbb{Z}_2$$

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### Definition

A $(t, m)$-configuration $(\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ is:

- *uniform* if pl is constant over $\mathcal{S}$.
- *planar* if the projective subspace generated by $\mathcal{V}$ is the whole $\mathbb{R}P^2$.

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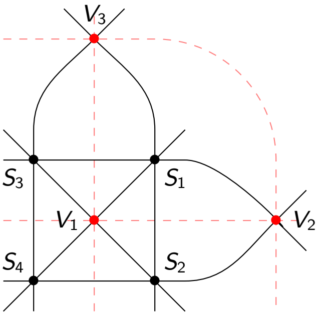
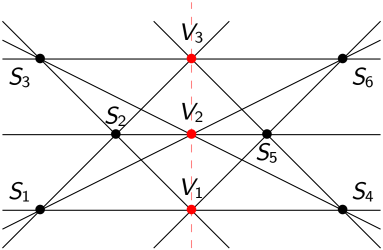



$$\mathrm{pl} : (S_1, S_2, S_3, S_4) \mapsto (1, 1, 1, 1) \in \mathbb{Z}_2^4$$

# Configurations of points

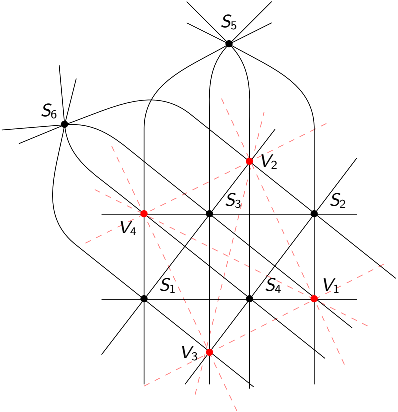
A non-planar and non-uniform $(3, m)$-configuration, $m \geq 3$ :




$$\mathrm{pl} : (S_1, \cdots, S_6) \to (\zeta, -\zeta, \zeta, -\zeta, \zeta, -\zeta) \in \mathbb{Z}_m^6$$

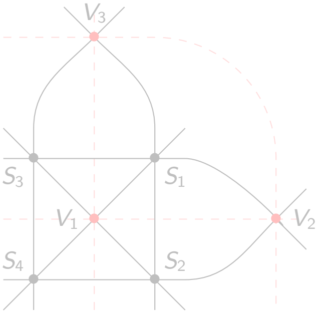
# Configurations of points

A planar and uniform $(4,2)$-configuration:



## Combinatorics of a configuration

COMBINATORICS: (nontrivial) collinearity relations between points $\mathcal{V} \sqcup \mathcal{S}$ in $\mathbb{R}P^2$.




$\{ \{V_1, S_1, S_4\}, \{V_1, S_2, S_3\}, \{V_2, S_1, S_3\}, \{V_2, S_2, S_4\}, \{V_3, S_1, S_2\}, \{V_3, S_3, S_4\} \}.$

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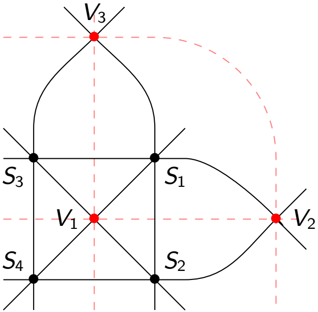


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The combinatorics of $\mathcal{C}$ is not invariant by deformation.

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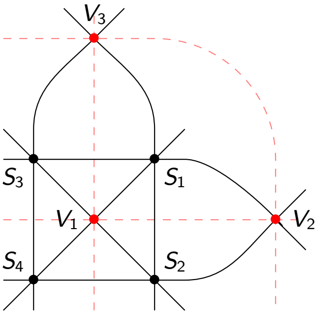


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This configuration is stable.

## Dual arrangement

We consider *dual real plane* $\check{\mathbb{R}P}^2 = \{L \mid L \subset \mathbb{R}P^2 \text{ droite}\}$.

- DUALITY : natural correspondence $(\cdot)^*$ between $\mathbb{R}P^2$ and $\check{\mathbb{R}P}^2$ *respecting incidence relations*: $P \in L \Longleftrightarrow L^* \in P^*$.

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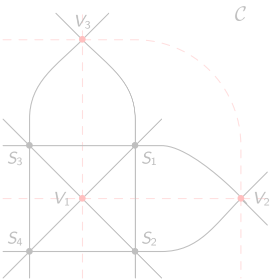


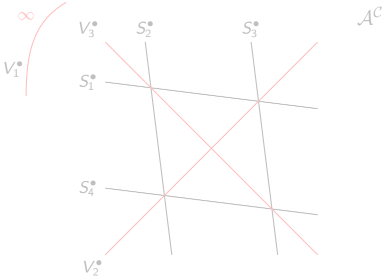

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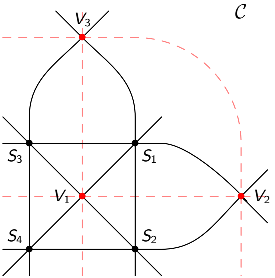


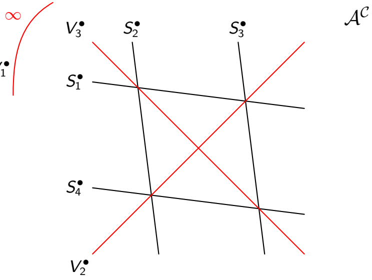

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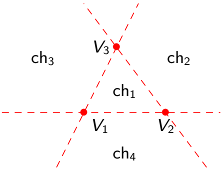
Part III

# Topology of arrangements and configurations

## Chamber weight and invariance

Take $\mathcal{C} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ a planar $(3, m)$-configuration: vertices $V_1$, $V_2$, $V_3$ define a partition of $\mathbb{R}P^2$ in 4 chambers




**Definition**

The *chamber weight* of $\mathcal{C}$ is the value

$$\tau(\mathcal{C}) = \sum_{S \in \mathcal{S} \cap \mathrm{ch}_i} \mathrm{pl}(S) \in \mathbb{Z}_m.$$

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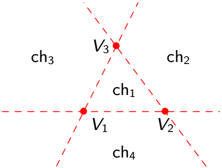


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*$\tau(\mathcal{C})$ does not depend on the choice of $\mathrm{ch}_i$, and*

$$\tau(\mathcal{C}) = \begin{cases} [0] \ \text{ or } \ [m/2] & \text{, if } m \text{ even} \\ [0] & \text{, if } m \text{ odd} \end{cases}$$

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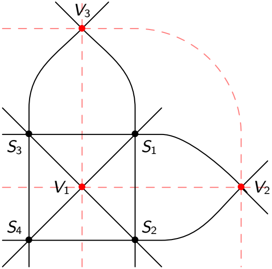
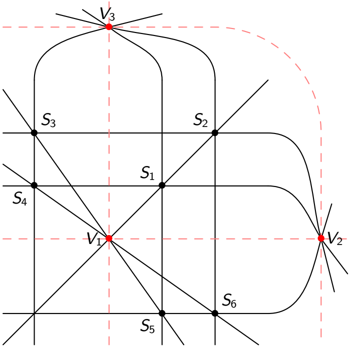



$\tau(\mathcal{C}_1) = 1$




$\tau(\mathcal{C}_2) = 0$

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**Theorem (Guerville-Ballé, ___ )**

*$\tau(\mathcal{C})$ is an invariant of the ordered topology of the dual arrangement $\mathcal{A}^{\mathcal{C}}$.*

**Corollary**

*Moreover, if $\mathcal{C}$ is stable and uniform, then $\tau(\mathcal{C})$ is a topological invariant of $\mathcal{A}^{\mathcal{C}}$.*

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## The Zariski pair game

QUESTION : Could be possible to construct Zariski pairs from $(3,2)$-configurations?

ZARISKI GAME IN $\mathbb{Z}_2$: Construct two $(3,2)$-configurations $\mathcal{C}_1$ and $\mathcal{C}_2$

1. Fix vertices $V_1, V_2, V_3$.

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## New real complexified Zariski pairs

Take $\alpha, \beta \in \{-1, 1\}$, let $\mathcal{C}_{\alpha,\beta} = (\mathcal{V}, \mathcal{S}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta}, \text{pl})$ be planar uniform (3, 2)-configurations defined over $\mathbb{Q}$ by:

$$\mathcal{V} = \{V_1, V_2, V_3\}, \quad \mathcal{S}_{\alpha,\beta} = \mathcal{S} \sqcup \mathcal{S}_\alpha \sqcup \mathcal{S}_\beta,$$

$$\mathcal{S} = \{S_1, \dots, S_4\}, \quad \mathcal{S}_\alpha = \{S_5^\alpha, S_6^\alpha, S_7^\alpha\}, \quad \mathcal{S}_\beta = \{S_8^\beta, S_9^\beta, S_{10}^\beta\}.$$

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## New real complexified Zariski pairs

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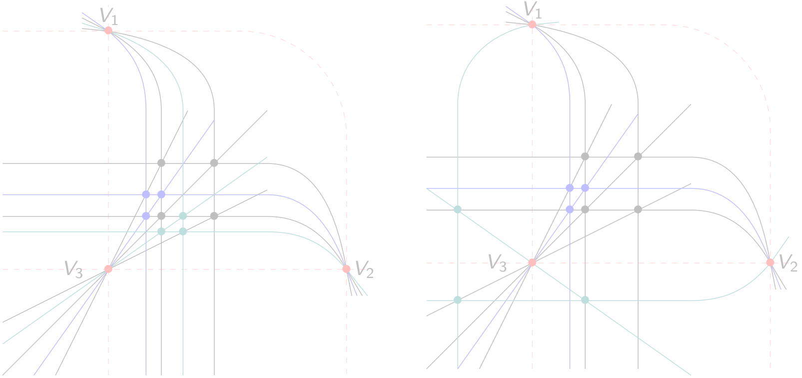


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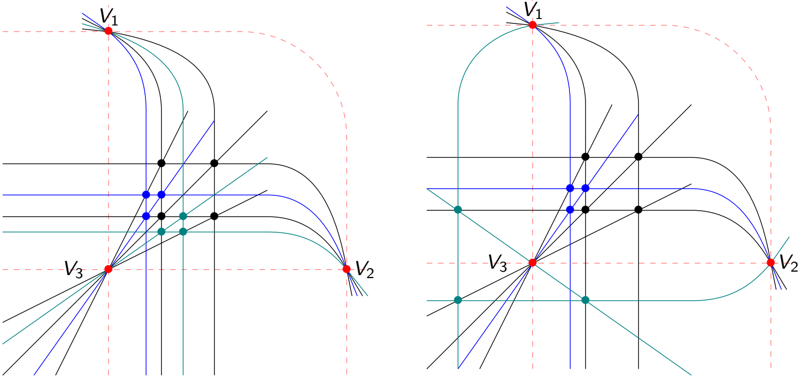


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## Moduli space and geometrical characterization

The *moduli space* $\Sigma_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ of $n$ lines:

$$\Sigma_{\mathcal{A}} = \{\mathcal{B} \in (\mathbb{C}P^2)^n \mid \mathcal{B} \sim_{\text{comb}} \mathcal{A}\} / \operatorname{PGL}_3(\mathbb{C}).$$

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*The moduli space $\Sigma$ of $\mathcal{A}^{\alpha,\beta}$ is formed by two connected components $\Sigma^0$ and $\Sigma^1$. Moreover,*

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Let $G_1 = \pi_1(\mathbb{C}P^2 \setminus \mathcal{A}^{1,1})$ and $G_2 = \pi_1(\mathbb{C}P^2 \setminus \mathcal{A}^{-1,1})$. We compute, using SAGE:

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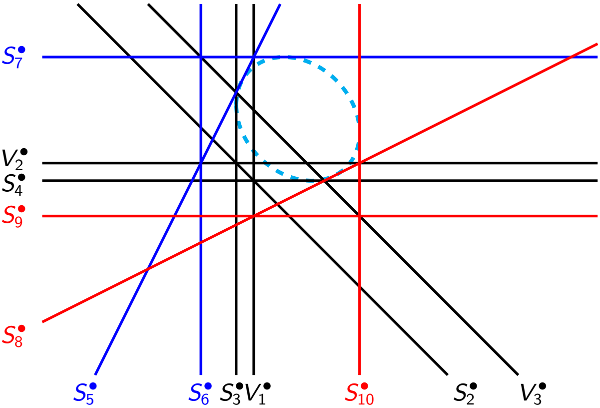


THANK YOU!