On the geometry of line arrangements and polynomial vector fields

Juan Viu-Sos

(A joined work with J. Cresson et B. Guerville-Ballé)

Functional Equations in LIMoges 2015
XLIM, University of Limoges, Faculty of Sciences and Techniques

22 mars 2015

Table of contents

Part I

Introduction

Logarithmic vector fields and logarithmic differential forms

- Introduced by K. Saito (~ 80) in order to study divisors in complex manifolds, generalizing P. Deligne. $\rightsquigarrow$ study of Gauss-Manin connection.
- Dynamical interpretation: "Logarithmic vector fields are holomorphic vector fields tangent to the smooth locus of a divisor D of X".

Logarithmic vector fields and logarithmic differential forms

- Introduced by K. Saito (~ 80) in order to study divisors in complex manifolds, generalizing P. Deligne. $\rightsquigarrow$ study of Gauss-Manin connection.
- Dynamical interpretation: "Logarithmic vector fields are holomorphic vector fields tangent to the smooth locus of a divisor D of X".
- Gives topological information of the complement $X \setminus D$.

Logarithmic vector fields and logarithmic differential forms

- Introduced by K. Saito (~ 80) in order to study divisors in complex manifolds, generalizing P. Deligne. $\rightsquigarrow$ study of Gauss-Manin connection.
- Dynamical interpretation: "Logarithmic vector fields are holomorphic vector fields tangent to the smooth locus of a divisor D of X".
- Gives topological information of the complement $X \setminus D$.
- A sufficient regular object becomes an *invariant set of the flow* of the logarithmic vector field.

Logarithmic vector fields and logarithmic differential forms

- Introduced by K. Saito (~ 80) in order to study divisors in complex manifolds, generalizing P. Deligne. $\rightsquigarrow$ study of Gauss-Manin connection.
- Dynamical interpretation: "Logarithmic vector fields are holomorphic vector fields tangent to the smooth locus of a divisor D of X".
- Gives topological information of the complement $X \setminus D$.
- A sufficient regular object becomes an *invariant set of the flow* of the logarithmic vector field.

$\rightsquigarrow$

Darboux polynomial of a polynomial differential system in $\mathbb{C}^2$.

Logarithmic vector fields and logarithmic differential forms

- Introduced by K. Saito (~ 80) in order to study divisors in complex manifolds, generalizing P. Deligne. $\rightsquigarrow$ study of Gauss-Manin connection.
- Dynamical interpretation: "Logarithmic vector fields are holomorphic vector fields tangent to the smooth locus of a divisor D of X".
- Gives topological information of the complement $X \setminus D$.
- A sufficient regular object becomes an *invariant set of the flow* of the logarithmic vector field.

$\rightsquigarrow$

Darboux polynomial of a polynomial differential system in $\mathbb{C}^2$.

- IDEA: Dynamical approach to affine/projective geometry for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
  - *Given a sufficiently regular geometric object O in $\mathbb{A}^n_{\mathbb{K}}$ or $\mathbb{P}^n_{\mathbb{K}}$, one can study the set denoted by $\mathcal{D}(O)$ of vector fields for which O is an invariant set.*
- K. Saito: analytic class $\rightsquigarrow$ H. Terao ($\sim$ 82) for line arrangements: it suffices to consider the algebraic class.

- IDEA: Dynamical approach to affine/projective geometry for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
  - *Given a sufficiently regular geometric object O in $\mathbb{A}^n_{\mathbb{K}}$ or $\mathbb{P}^n_{\mathbb{K}}$, one can study the set denoted by $\mathcal{D}(O)$ of vector fields for which O is an invariant set.*
- K. Saito: analytic class $\rightsquigarrow$ H. Terao ($\sim$ 82) for line arrangements: it suffices to consider the algebraic class.
- Classical problems of plane differential systems simplifying to *algebraic* vector fields fixing *configurations of algebraic curves.*

- IDEA: Dynamical approach to affine/projective geometry for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
  - *Given a sufficiently regular geometric object O in $\mathbb{A}^n_{\mathbb{K}}$ or $\mathbb{P}^n_{\mathbb{K}}$, one can study the set denoted by $\mathcal{D}(O)$ of vector fields for which O is an invariant set.*
- K. Saito: analytic class $\rightsquigarrow$ H. Terao ($\sim$ 82) for line arrangements: it suffices to consider the algebraic class.
- Classical problems of plane differential systems simplifying to *algebraic* vector fields fixing *configurations of algebraic curves*.
  - Dulac's conjecture: *"There is a finite number of (algebraic) limit cycles of plane polynomial vector fields"*.
  - Algebraic Hilbert's 16th Problem: *"Is there a bound C on the number of algebraic limit cycles of a polynomial vector field such that $C \leq d^q$ for $q > 0$"*.

- IDEA: Dynamical approach to affine/projective geometry for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
  - *Given a sufficiently regular geometric object O in $\mathbb{A}^n_{\mathbb{K}}$ or $\mathbb{P}^n_{\mathbb{K}}$, one can study the set denoted by $\mathcal{D}(O)$ of vector fields for which O is an invariant set.*
- K. Saito: analytic class $\rightsquigarrow$ H. Terao ($\sim$ 82) for line arrangements: it suffices to consider the algebraic class.
- Classical problems of plane differential systems simplifying to *algebraic* vector fields fixing *configurations of algebraic curves*.
  - Dulac's conjecture: *"There is a finite number of (algebraic) limit cycles of plane polynomial vector fields"*.
  - Algebraic Hilbert's 16th Problem: *"Is there a bound C on the number of algebraic limit cycles of a polynomial vector field such that $C \leq d^q$ for $q > 0$"*.

PART II

A DYNAMICAL APPROACH TO LINE ARRANGEMENTS

"Combinatorics of line arrangements and dynamics of polynomial vector fields."
arXiv:1412.0137, 14 pages, with B. Guerville-Ballé. (Submitted)

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Definition

An *affine* (resp. *projective*) *line arrangement* $\mathcal{A}$ is a finite collection $\{L_1, \dots, L_n\}$ of lines in $\mathbb{A}^2_{\mathbb{K}}$ (resp. $\mathbb{P}^2_{\mathbb{K}}$).

- DEFINING POLYNOMIAL: $\mathcal{Q}_{\mathcal{A}} = \prod_{L \in \mathcal{A}} \alpha_L$ where α_L is an affine (resp. linear) form verifying $L = \ker \alpha_L$.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Definition

An *affine* (resp. *projective*) *line arrangement* $\mathcal{A}$ is a finite collection $\{L_1, \ldots, L_n\}$ of lines in $\mathbb{A}^2_{\mathbb{K}}$ (resp. $\mathbb{P}^2_{\mathbb{K}}$).

- DEFINING POLYNOMIAL: $\mathcal{Q}_{\mathcal{A}} = \prod_{L \in \mathcal{A}} \alpha_L$ where α_L is an affine (resp. linear) form verifying $L = \ker \alpha_L$.
- COMBINATORIAL DATA: encoded in the *intersection poset*

$$L(\mathcal{A}) = \{\emptyset \neq L_i \cap L_j \mid L_i, L_j \in \mathcal{A}\} \cup \mathcal{A}$$

(partially ordered by reverse inclusion of subsets).

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

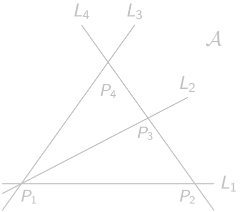
Definition

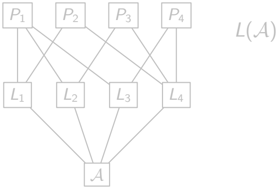
An *affine* (resp. *projective*) *line arrangement* $\mathcal{A}$ is a finite collection $\{L_1, \dots, L_n\}$ of lines in $\mathbb{A}^2_{\mathbb{K}}$ (resp. $\mathbb{P}^2_{\mathbb{K}}$).

- DEFINING POLYNOMIAL: $\mathcal{Q}_{\mathcal{A}} = \prod_{L \in \mathcal{A}} \alpha_L$ where α_L is an affine (resp. linear) form verifying $L = \ker \alpha_L$.
- COMBINATORIAL DATA: encoded in the *intersection poset*

$$L(\mathcal{A}) = \{\emptyset \neq L_i \cap L_j \mid L_i, L_j \in \mathcal{A}\} \cup \mathcal{A}$$

(partially ordered by reverse inclusion of subsets).





Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

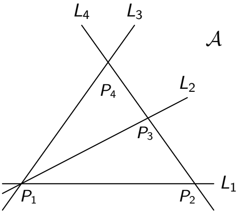
Definition

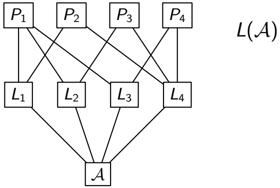
An *affine* (resp. *projective*) *line arrangement* $\mathcal{A}$ is a finite collection $\{L_1, \dots, L_n\}$ of lines in $\mathbb{A}^2_{\mathbb{K}}$ (resp. $\mathbb{P}^2_{\mathbb{K}}$).

- DEFINING POLYNOMIAL: $\mathcal{Q}_{\mathcal{A}} = \prod_{L \in \mathcal{A}} \alpha_L$ where α_L is an affine (resp. linear) form verifying $L = \ker \alpha_L$.
- COMBINATORIAL DATA: encoded in the *intersection poset*

$$L(\mathcal{A}) = \{\emptyset \neq L_i \cap L_j \mid L_i, L_j \in \mathcal{A}\} \cup \mathcal{A}$$

(partially ordered by reverse inclusion of subsets).





What is the influence of the combinatorics on the embedding of $\mathcal{A}$?

- Topological invariants:
  - Exterior: $E_{\mathcal{A}} = \mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{A}$ or $\mathbb{P}^2_{\mathbb{K}} \setminus \mathcal{A}$.
  - Fundamental group: $\pi_1(E_{\mathcal{A}})$ (interesting when $\mathbb{K} = \mathbb{C}$)
  - Milnor fiber: $\mathcal{F}_{\mathcal{A}} = \mathcal{Q}_{\mathcal{A}}^{-1}(1)$

What is the influence of the combinatorics on the embedding of $\mathcal{A}$?

- Topological invariants:
  - Exterior: $E_{\mathcal{A}} = \mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{A}$ or $\mathbb{P}^2_{\mathbb{K}} \setminus \mathcal{A}$.
  - Fundamental group: $\pi_1(E_{\mathcal{A}})$ (interesting when $\mathbb{K} = \mathbb{C}$)
  - Milnor fiber: $\mathcal{F}_{\mathcal{A}} = \mathcal{Q}_{\mathcal{A}}^{-1}(1)$
- Algebraic geometrical objects:
  - Cohomological algebras: $H^\bullet(E_{\mathcal{A}})$, $H^\bullet(\mathcal{F}_{\mathcal{A}}), \dots$
  - Logarithmic differential forms: $\Omega^\bullet(\log \mathcal{A})$

  $\rightsquigarrow$ Logarithmic vector fields: $\mathcal{D}(\mathcal{A}) = \left(\Omega^1(\log \mathcal{A})\right)^*$

What is the influence of the combinatorics on the embedding of $\mathcal{A}$?

- Topological invariants:
  - Exterior: $E_{\mathcal{A}} = \mathbb{A}^2_{\mathbb{K}} \setminus \mathcal{A}$ or $\mathbb{P}^2_{\mathbb{K}} \setminus \mathcal{A}$.
  - Fundamental group: $\pi_1(E_{\mathcal{A}})$ (interesting when $\mathbb{K} = \mathbb{C}$)
  - Milnor fiber: $\mathcal{F}_{\mathcal{A}} = \mathcal{Q}_{\mathcal{A}}^{-1}(1)$
- Algebraic geometrical objects:
  - Cohomological algebras: $H^\bullet(E_{\mathcal{A}})$, $H^\bullet(\mathcal{F}_{\mathcal{A}}), \ldots$
  - Logarithmic differential forms: $\Omega^\bullet(\log \mathcal{A})$
  - $\rightsquigarrow$ Logarithmic vector fields: $\mathcal{D}(\mathcal{A}) = \big(\Omega^1(\log \mathcal{A})\big)^*$

Let $S = \mathbb{K}[x, y]$, define the *algebra of* $\mathbb{K}$*–derivations of* S as

$$\mathrm{Der}_{\mathbb{K}}(S) = \{\chi : S \to S \ \mathbb{K} - \text{linear} \mid \chi(fg) = \chi(f)g + f\chi(g), \forall f, g \in S\}$$

A derivation can be viewed as a *polynomial vector field* in the plane

$$\chi = P\partial_x + Q\partial_y, \quad \text{where } P, Q \in S.$$

Definition

The *S-module of logarithmic derivations of* $\mathcal{A}$

$$\mathcal{D}(\mathcal{A}) = \{\chi \in \mathrm{Der}_{\mathbb{K}}(S) \mid \chi\mathcal{Q}_{\mathcal{A}} \in \mathcal{I}_{\mathcal{Q}_{\mathcal{A}}}\}$$

where $\mathcal{I}_{\mathcal{Q}_{\mathcal{A}}}$ is the ideal in S generated by $\mathcal{Q}_{\mathcal{A}}$.

Let $S = \mathbb{K}[x, y]$, define the *algebra of* $\mathbb{K}$*–derivations of* S as

$$\mathrm{Der}_{\mathbb{K}}(S) = \{\chi : S \to S \ \mathbb{K} - \text{linear} \mid \chi(fg) = \chi(f)g + f\chi(g), \forall f, g \in S\}$$

A derivation can be viewed as a *polynomial vector field* in the plane

$$\chi = P\partial_x + Q\partial_y, \quad \text{where } P, Q \in S.$$

Definition

The *S-module of logarithmic derivations of* $\mathcal{A}$

$$\mathcal{D}(\mathcal{A}) = \{\chi \in \mathrm{Der}_{\mathbb{K}}(S) \mid \chi \mathcal{Q}_{\mathcal{A}} \in \mathcal{I}_{\mathcal{Q}_{\mathcal{A}}}\}$$

where $\mathcal{I}_{\mathcal{Q}_{\mathcal{A}}}$ is the ideal in S generated by $\mathcal{Q}_{\mathcal{A}}$.

Proposition

$$\chi \in \mathcal{D}(\mathcal{A}) \iff \exists K \in S \text{ such that } \chi \mathcal{Q}_{\mathcal{A}} = K \mathcal{Q}_{\mathcal{A}}$$
$$\iff \mathcal{A} \subset \mathbb{A}^2_{\mathbb{K}} \text{ is an invariant set of } \chi$$

Let $S = \mathbb{K}[x, y]$, define the *algebra of* $\mathbb{K}$*–derivations of* S as

$$\mathrm{Der}_{\mathbb{K}}(S) = \{\chi : S \to S \ \mathbb{K} - \text{linear} \mid \chi(fg) = \chi(f)g + f\chi(g), \forall f, g \in S\}$$

A derivation can be viewed as a *polynomial vector field* in the plane

$$\chi = P\partial_x + Q\partial_y, \quad \text{where } P, Q \in S.$$

Definition

The *S-module of logarithmic derivations of* $\mathcal{A}$

$$\mathcal{D}(\mathcal{A}) = \{\chi \in \mathrm{Der}_{\mathbb{K}}(S) \mid \chi\mathcal{Q}_{\mathcal{A}} \in \mathcal{I}_{\mathcal{Q}_{\mathcal{A}}}\}$$

where $\mathcal{I}_{\mathcal{Q}_{\mathcal{A}}}$ is the ideal in S generated by $\mathcal{Q}_{\mathcal{A}}$.

Proposition

$$\chi \in \mathcal{D}(\mathcal{A}) \iff \exists K \in S \textit{ such that } \chi\mathcal{Q}_{\mathcal{A}} = K\mathcal{Q}_{\mathcal{A}}$$

$$\iff \mathcal{A} \subset \mathbb{A}^2_{\mathbb{K}} \textit{ is an invariant set of } \chi$$

We can define an *ascending filtration by degree* of $\mathcal{D}(\mathcal{A})$ by vector spaces:

$$\mathcal{D}(\mathcal{A}) = \bigcup_{d\in\mathbb{N}} \mathcal{F}_d\mathcal{D}(\mathcal{A}) \quad \text{with} \quad \mathcal{F}_d\mathcal{D}(\mathcal{A}) \subset \mathcal{F}_{d+1}\mathcal{D}(\mathcal{A})$$

where

$$\mathcal{F}_d\mathcal{D}(\mathcal{A}) = \{P\partial_x + Q\partial_y \in \mathcal{D}(\mathcal{A}) \mid \deg P, \deg Q \leq d\}$$

Definition

We denote by

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{F}_d\mathcal{D}(\mathcal{A}) \setminus \mathcal{F}_{d-1}\mathcal{D}(\mathcal{A})$$

the set of *polynomial vector fields of degree d fixing $\mathcal{A}$.*

We can define an *ascending filtration by degree* of $\mathcal{D}(\mathcal{A})$ by vector spaces:

$$\mathcal{D}(\mathcal{A}) = \bigcup_{d\in\mathbb{N}} \mathcal{F}_d\mathcal{D}(\mathcal{A}) \quad \text{with} \quad \mathcal{F}_d\mathcal{D}(\mathcal{A}) \subset \mathcal{F}_{d+1}\mathcal{D}(\mathcal{A})$$

where

$$\mathcal{F}_d\mathcal{D}(\mathcal{A}) = \{P\partial_x + Q\partial_y \in \mathcal{D}(\mathcal{A}) \mid \deg P, \deg Q \leq d\}$$

Definition

We denote by

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{F}_d\mathcal{D}(\mathcal{A}) \setminus \mathcal{F}_{d-1}\mathcal{D}(\mathcal{A})$$

the set of *polynomial vector fields of degree d fixing* $\mathcal{A}$.

Efficiently characterization of line arrangements as invariant sets of polynomial vector fields.

⇝

When $\chi \in \mathrm{Der}_{\mathbb{K}}(S)$ posses a finite family of invariant lines?

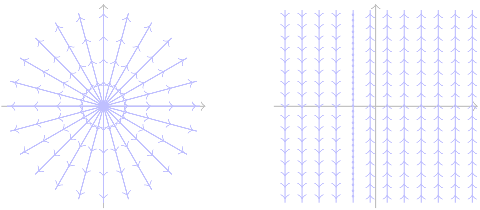


Figure: Phase portraits of $\chi_c = x\partial_x + y\partial_y$ and $\chi_p = (x+1)\partial_y$.

Efficiently characterization of line arrangements as invariant sets of polynomial vector fields.

⇝

When $\chi \in \mathrm{Der}_{\mathbb{K}}(S)$ posses a finite family of invariant lines?

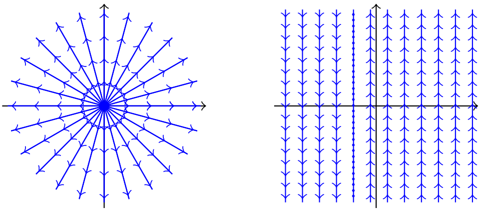


Figure: Phase portraits of $\chi_c = x\partial_x + y\partial_y$ and $\chi_p = (x+1)\partial_y$.

Definition

Let χ be a polynomial vector field in the plane. We said that a line arrangement $\mathcal{A}$ is *maximal fixed by* χ if any invariant line L by χ belongs to $\mathcal{A}$.

Definition

We said that χ *fixes only a finite set of lines* if there exists a maximal arrangement fixed by χ. Conversely, we said that χ *fixes an infinity of lines* if there is no such maximal line arrangement.

Definition

Let χ be a polynomial vector field in the plane. We said that a line arrangement $\mathcal{A}$ is *maximal fixed by* χ if any invariant line L by χ belongs to $\mathcal{A}$.

Definition

We said that χ *fixes only a finite set of lines* if there exists a maximal arrangement fixed by χ. Conversely, we said that χ *fixes an infinity of lines* if there is no such maximal line arrangement.

Following this notion, we are interested to study the partition

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{D}_d^f(\mathcal{A}) \cup \mathcal{D}_d^\infty(\mathcal{A})$$

and the number $d_f(\mathcal{A}) = \min\{d \in \mathbb{N} \mid \mathcal{D}_d^f(\mathcal{A}) \neq \emptyset\}$.

Definition

Let χ be a polynomial vector field in the plane. We said that a line arrangement $\mathcal{A}$ is *maximal fixed by* χ if any invariant line L by χ belongs to $\mathcal{A}$.

Definition

We said that χ *fixes only a finite set of lines* if there exists a maximal arrangement fixed by χ. Conversely, we said that χ *fixes an infinity of lines* if there is no such maximal line arrangement.

Following this notion, we are interested to study the partition

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{D}_d^f(\mathcal{A}) \cup \mathcal{D}_d^\infty(\mathcal{A})$$

and the number $d_f(\mathcal{A}) = \min\{d \in \mathbb{N} \mid \mathcal{D}_d^f(\mathcal{A}) \neq \emptyset\}$.

Characterization of $\mathcal{D}_d^\infty(\mathcal{A})$

Theorem

If $\chi \in \mathcal{D}_d^\infty(\mathcal{A})$, then χ belongs to one of these classes of vector fields

1. NULL.
2. RADIAL: *there exist a point $(x_0, y_0) \in \mathbb{A}^2_{\mathbb{K}}$ such that*

$$(y_0 - y, x - x_0) \perp \chi(x, y), \quad \text{for any } (x, y) \in \mathbb{A}^2_{\mathbb{K}}.$$

3. PARALLEL: *there exist a vector $\vec{v} \in \mathbb{A}^2_{\mathbb{K}}$ such that*

$$\vec{v} \parallel \chi(x, y), \quad \text{for any } (x, y) \in \mathbb{A}^2_{\mathbb{K}}.$$

Characterization of $\mathcal{D}_d^\infty(\mathcal{A})$

Theorem

If $\chi \in \mathcal{D}_d^\infty(\mathcal{A})$, then χ belongs to one of these classes of vector fields

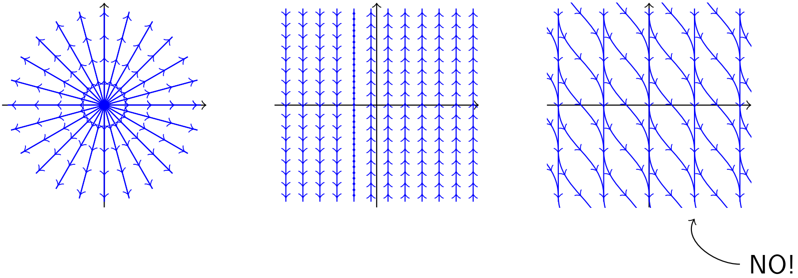
1. NULL.
2. RADIAL: *there exist a point $(x_0, y_0) \in \mathbb{A}^2_{\mathbb{K}}$ such that*

$$(y_0 - y, x - x_0) \perp \chi(x, y), \quad \textit{for any } (x, y) \in \mathbb{A}^2_{\mathbb{K}}.$$

3. PARALLEL: *there exist a vector $\vec{v} \in \mathbb{A}^2_{\mathbb{K}}$ such that*

$$\vec{v} \parallel \chi(x, y), \quad \textit{for any } (x, y) \in \mathbb{A}^2_{\mathbb{K}}.$$

Characterization of $\mathcal{D}_d^{\infty}(\mathcal{A})$



Influence of the combinatorics in $\mathcal{D}_d^\infty(\mathcal{A})$

Define the combinatorial data:

- $|\mathcal{A}|$ the number of lines in $\mathcal{A}$.
- $m(\mathcal{A})$ the maximal multiplicity of the singularities in $\mathcal{A}$.
- $p(\mathcal{A})$ the maximal number of parallel lines in $\mathcal{A}$.

Theorem

Let $0 \neq \chi \in \mathcal{D}_d(\mathcal{A})$:

1. *If $d < m(\mathcal{A}) - 1$ then χ is a radial vector field.*
2. *If $d < p(\mathcal{A})$ then χ is a parallel vector field.*

Influence of the combinatorics in $\mathcal{D}_d^\infty(\mathcal{A})$

Define the combinatorial data:

- $|\mathcal{A}|$ the number of lines in $\mathcal{A}$.
- $m(\mathcal{A})$ the maximal multiplicity of the singularities in $\mathcal{A}$.
- $p(\mathcal{A})$ the maximal number of parallel lines in $\mathcal{A}$.

Theorem

Let $0 \neq \chi \in \mathcal{D}_d(\mathcal{A})$:

1. *If $d < m(\mathcal{A}) - 1$ then χ is a radial vector field.*
2. *If $d < p(\mathcal{A})$ then χ is a parallel vector field.*

Corollary (1)

Define $\nu_\infty(\mathcal{A}) = \max\{m(\mathcal{A}) - 1, p(\mathcal{A})\}$. Then

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{D}_d^\infty(\mathcal{A}), \quad \forall d < \nu_\infty(\mathcal{A})$$

Influence of the combinatorics in $\mathcal{D}_d^{\infty}(\mathcal{A})$

Define the combinatorial data:

- $|\mathcal{A}|$ the number of lines in $\mathcal{A}$.
- $m(\mathcal{A})$ the maximal multiplicity of the singularities in $\mathcal{A}$.
- $p(\mathcal{A})$ the maximal number of parallel lines in $\mathcal{A}$.

Theorem

Let $0 \neq \chi \in \mathcal{D}_d(\mathcal{A})$*:*

1. *If* $d < m(\mathcal{A}) - 1$ *then* χ *is a radial vector field.*
2. *If* $d < p(\mathcal{A})$ *then* χ *is a parallel vector field.*

Corollary (1)

Define $\nu_\infty(\mathcal{A}) = \max\{m(\mathcal{A}) - 1, p(\mathcal{A})\}$*. Then*

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{D}_d^{\infty}(\mathcal{A}), \quad \forall d < \nu_\infty(\mathcal{A})$$

Influence of the combinatorics in $\mathcal{D}_d^{\infty}(\mathcal{A})$

Corollary (2)

We have $d_f(\mathcal{A}) \geq \nu_{\infty}(\mathcal{A})$.

Influence of the combinatorics in $\mathcal{D}_d^\infty(\mathcal{A})$

Corollary (2)

We have $d_f(\mathcal{A}) \geq \nu_\infty(\mathcal{A})$.

Corollary (3)

Define $\eta_\infty(\mathcal{A}) = \min\{m(\mathcal{A})-1, p(\mathcal{A})\}$. *Then*

$$\mathcal{D}_d(\mathcal{A}) = \emptyset, \quad \forall 0 < d < \eta_\infty(\mathcal{A})$$

Influence of the combinatorics in $\mathcal{D}_d^\infty(\mathcal{A})$

Corollary (2)

We have $d_f(\mathcal{A}) \geq \nu_\infty(\mathcal{A})$.

Corollary (3)

Define $\eta_\infty(\mathcal{A}) = \min\{m(\mathcal{A}) - 1, p(\mathcal{A})\}$. *Then*

$$\mathcal{D}_d(\mathcal{A}) = \emptyset, \quad \forall 0 < d < \eta_\infty(\mathcal{A})$$

Influence of the combinatorics in $\mathcal{D}_d^f(\mathcal{A})$

Theorem

1. *The minimal degree of a radial vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - m(\mathcal{A}) + 1$.
2. *The minimal degree of a parallel vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - p(\mathcal{A})$.

Influence of the combinatorics in $\mathcal{D}_d^f(\mathcal{A})$

Theorem

1. *The minimal degree of a radial vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - m(\mathcal{A}) + 1$.
2. *The minimal degree of a parallel vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - p(\mathcal{A})$.

Corollary (1)

Define $\nu_f(\mathcal{A}) = \min\{|\mathcal{A}| - m(\mathcal{A}) + 1, |\mathcal{A}| - p(\mathcal{A})\}$. *Then*

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{D}_d^f(\mathcal{A}), \quad \forall 0 < d < \nu_f(\mathcal{A})$$

Influence of the combinatorics in $\mathcal{D}_d^f(\mathcal{A})$

Theorem

1. *The minimal degree of a radial vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - m(\mathcal{A}) + 1$.
2. *The minimal degree of a parallel vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - p(\mathcal{A})$.

Corollary (1)

Define $\nu_f(\mathcal{A}) = \min\{|\mathcal{A}| - m(\mathcal{A}) + 1, |\mathcal{A}| - p(\mathcal{A})\}$. *Then*

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{D}_d^f(\mathcal{A}), \quad \forall 0 < d < \nu_f(\mathcal{A})$$

Corollary (2)

Let $\nu(\mathcal{A}) = \min\{\nu_\infty(\mathcal{A}), \nu_f(\mathcal{A})\}$. *Then*

$$\mathcal{D}_d(\mathcal{A}) = \emptyset, \quad \forall 0 < d < \nu(\mathcal{A})$$

Influence of the combinatorics in $\mathcal{D}_d^f(\mathcal{A})$

Theorem

1. *The minimal degree of a radial vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - m(\mathcal{A}) + 1$.
2. *The minimal degree of a parallel vector field in* $\mathcal{D}(\mathcal{A})$ *is* $|\mathcal{A}| - p(\mathcal{A})$.

Corollary (1)

Define $\nu_f(\mathcal{A}) = \min\{|\mathcal{A}| - m(\mathcal{A}) + 1, |\mathcal{A}| - p(\mathcal{A})\}$. *Then*

$$\mathcal{D}_d(\mathcal{A}) = \mathcal{D}_d^f(\mathcal{A}), \quad \forall 0 < d < \nu_f(\mathcal{A})$$

Corollary (2)

Let $\nu(\mathcal{A}) = \min\{\nu_\infty(\mathcal{A}), \nu_f(\mathcal{A})\}$. *Then*

$$\mathcal{D}_d(\mathcal{A}) = \emptyset, \quad \forall 0 < d < \nu(\mathcal{A})$$

- STRONG COMBINATORICS: The poset $L(\mathcal{A})$.
- WEAK COMBINATORICS: The tuple $(|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}, \mathcal{P}_{\mathcal{A}})$ where $\mathcal{S}_{\mathcal{A}} = (s_m)_{m\in\mathbb{N}}$ and $\mathcal{P}_{\mathcal{A}} = (p_m)_{m\in\mathbb{N}}$ with:
  - s_m being the number of singularities in $\mathcal{A}$ of multiplicity m.
  - p_m being the number of families of exactly m lines in $\mathcal{A}$ which are parallel.

$d_f(\mathcal{A})$ is determined by the weak/strong combinatorics of $\mathcal{A}$?

- STRONG COMBINATORICS: The poset $L(\mathcal{A})$.
- WEAK COMBINATORICS: The tuple $(|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}, \mathcal{P}_{\mathcal{A}})$ where $\mathcal{S}_{\mathcal{A}} = (s_m)_{m\in\mathbb{N}}$ and $\mathcal{P}_{\mathcal{A}} = (p_m)_{m\in\mathbb{N}}$ with:
  - s_m being the number of singularities in $\mathcal{A}$ of multiplicity m.
  - p_m being the number of families of exactly m lines in $\mathcal{A}$ which are parallel.

$d_f(\mathcal{A})$ is determined by the weak/strong combinatorics of $\mathcal{A}$?

- STRONG COMBINATORICS: The poset $L(\mathcal{A})$.
- WEAK COMBINATORICS: The tuple $(|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}, \mathcal{P}_{\mathcal{A}})$ where $\mathcal{S}_{\mathcal{A}} = (s_m)_{m\in\mathbb{N}}$ and $\mathcal{P}_{\mathcal{A}} = (p_m)_{m\in\mathbb{N}}$ with:
  - s_m being the number of singularities in $\mathcal{A}$ of multiplicity m.
  - p_m being the number of families of exactly m lines in $\mathcal{A}$ which are parallel.

$d_f(\mathcal{A})$ is determined by the weak/strong combinatorics of $\mathcal{A}$?

NO! Two explicit counter-examples.

- STRONG COMBINATORICS: The poset $L(\mathcal{A})$.
- WEAK COMBINATORICS: The tuple $(|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}, \mathcal{P}_{\mathcal{A}})$ where $\mathcal{S}_{\mathcal{A}} = (s_m)_{m\in\mathbb{N}}$ and $\mathcal{P}_{\mathcal{A}} = (p_m)_{m\in\mathbb{N}}$ with:
  - s_m being the number of singularities in $\mathcal{A}$ of multiplicity m.
  - p_m being the number of families of exactly m lines in $\mathcal{A}$ which are parallel.

$d_f(\mathcal{A})$ is determined by the weak/strong combinatorics of $\mathcal{A}$?

NO! Two explicit counter-examples.

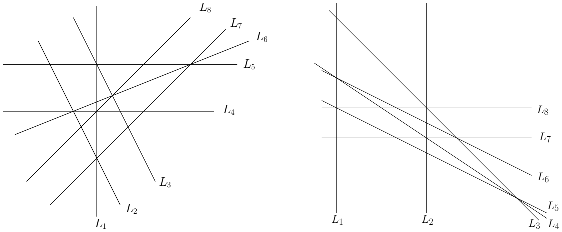


Figure: The Pappus and Non-Pappus arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$.

Same weak combinatorics: 8 lines, 6 triple points, 7 double points and 3 couples of parallel lines.

Using a suite of functions coded in Sage in order to study the filtration of $\mathcal{D}(\mathcal{A})$ and using the previous results:

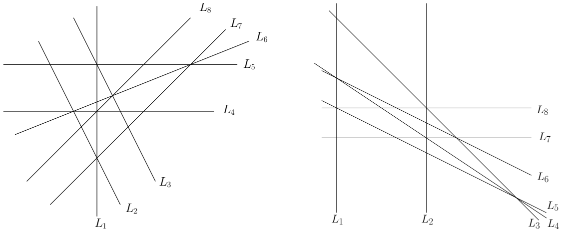


Figure: The Pappus and Non-Pappus arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$.

Same weak combinatorics: 8 lines, 6 triple points, 7 double points and 3 couples of parallel lines.

Using a suite of functions coded in Sage in order to study the filtration of $\mathcal{D}(\mathcal{A})$ and using the previous results:

Proposition

1. $\dim \mathcal{F}_3(\mathcal{P}_1) = 0$, $\dim \mathcal{F}_4(\mathcal{P}_1) > 0$ *and* $d_f(\mathcal{P}_1) = 4$.
2. $\dim \mathcal{F}_4(\mathcal{P}_2) = 0$, $\dim \mathcal{F}_5(\mathcal{P}_2) > 0$ *and* $d_f(\mathcal{P}_2) = 5$.

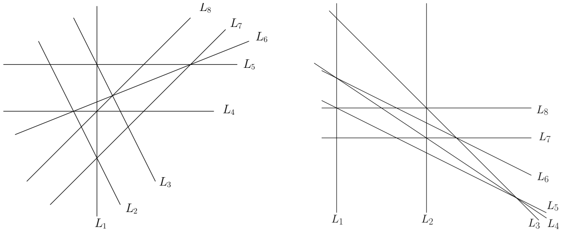


Figure: The Pappus and Non-Pappus arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$.

Same weak combinatorics: 8 lines, 6 triple points, 7 double points and 3 couples of parallel lines.

Using a suite of functions coded in Sage in order to study the filtration of $\mathcal{D}(\mathcal{A})$ and using the previous results:

Proposition

1. $\dim \mathcal{F}_3(\mathcal{P}_1) = 0$, $\dim \mathcal{F}_4(\mathcal{P}_1) > 0$ *and* $d_f(\mathcal{P}_1) = 4$.
2. $\dim \mathcal{F}_4(\mathcal{P}_2) = 0$, $\dim \mathcal{F}_5(\mathcal{P}_2) > 0$ *and* $d_f(\mathcal{P}_2) = 5$.

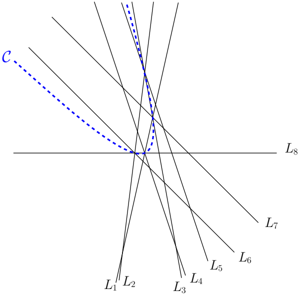


Figure: The Ziegler arrangement $\mathcal{Z}_1$.

- The 4 singularities of maximal multiplicity 3 in $\mathcal{Z}_1$ are contained in a conic $\mathcal{C}$.
- We construct a line arrangement $\mathcal{Z}_2$ by a perturbation displacing the triple point $L_1 \cap L_3 \cap L_7$ outside the conic, and such that $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$.

Again, studying the filtration with Sage and with the previous results:

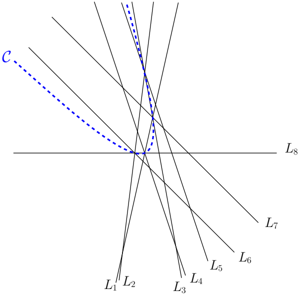


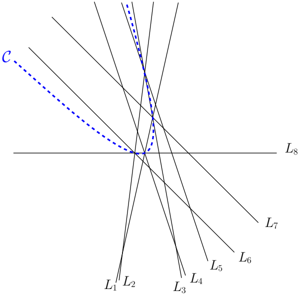
Figure: The Ziegler arrangement $\mathcal{Z}_1$.

- The 4 singularities of maximal multiplicity 3 in $\mathcal{Z}_1$ are contained in a conic $\mathcal{C}$.
- We construct a line arrangement $\mathcal{Z}_2$ by a perturbation displacing the triple point $L_1 \cap L_3 \cap L_7$ outside the conic, and such that $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$.

Again, studying the filtration with Sage and with the previous results:

Proposition

1. $\dim \mathcal{F}_4(\mathcal{Z}_1) = 0$, $\dim \mathcal{F}_5(\mathcal{Z}_1) > 0$ *and* $d_f(\mathcal{Z}_1) = 5$.
2. $\dim \mathcal{F}_5(\mathcal{Z}_2) = 0$, $\dim \mathcal{F}_6(\mathcal{Z}_2) > 0$ *and* $d_f(\mathcal{Z}_2) = 6$.



- The 4 singularities of maximal multiplicity 3 in $\mathcal{Z}_1$ are contained in a conic $\mathcal{C}$.
- We construct a line arrangement $\mathcal{Z}_2$ by a perturbation displacing the triple point $L_1 \cap L_3 \cap L_7$ outside the conic, and such that $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$.

Figure: The Ziegler arrangement $\mathcal{Z}_1$.

Again, studying the filtration with Sage and with the previous results:

Proposition

1. $\dim \mathcal{F}_4(\mathcal{Z}_1) = 0$, $\dim \mathcal{F}_5(\mathcal{Z}_1) > 0$ *and* $d_f(\mathcal{Z}_1) = 5$.
2. $\dim \mathcal{F}_5(\mathcal{Z}_2) = 0$, $\dim \mathcal{F}_6(\mathcal{Z}_2) > 0$ *and* $d_f(\mathcal{Z}_2) = 6$.

Part III

Conclusions and perspectives

In general, for hyperplane arrangements in $\mathbb{A}^n_{\mathbb{K}}$, the module of logarithmic derivations is not completely determined by the combinatorics of the arrangement.

Terao's Conjecture ('80s)

Consider $\mathcal{A}, \mathcal{A}' \subset \mathbb{A}^n_{\mathbb{K}}$ be two hyperplane arrangements containing the origin such that $L(\mathcal{A}) \simeq L(\mathcal{A}')$:

In general, for hyperplane arrangements in $\mathbb{A}^n_{\mathbb{K}}$, the module of logarithmic derivations is not completely determined by the combinatorics of the arrangement.

Terao's Conjecture ('80s)

Consider $\mathcal{A}, \mathcal{A}' \subset \mathbb{A}^n_{\mathbb{K}}$ be two hyperplane arrangements containing the origin such that $L(\mathcal{A}) \simeq L(\mathcal{A}')$:

1. In this setting: $\mathcal{D}(\mathcal{A}) = \bigoplus_{d \in \mathbb{N}} \mathcal{D}_d(\mathcal{A})$ by homogeneous components.
2. We said that $\mathcal{A}$ is *free* if $\mathcal{D}(\mathcal{A})$ is a free S-module.

In general, for hyperplane arrangements in $\mathbb{A}_{\mathbb{K}}^n$, the module of logarithmic derivations is not completely determined by the combinatorics of the arrangement.

Terao's Conjecture ('80s)

Consider $\mathcal{A}, \mathcal{A}' \subset \mathbb{A}_{\mathbb{K}}^n$ be two hyperplane arrangements containing the origin such that $L(\mathcal{A}) \simeq L(\mathcal{A}')$:

1. In this setting: $\mathcal{D}(\mathcal{A}) = \bigoplus_{d\in\mathbb{N}} \mathcal{D}_d(\mathcal{A})$ by homogeneous components.
2. We said that $\mathcal{A}$ is *free* if $\mathcal{D}(\mathcal{A})$ is a free S-module.

Conjecture (Weak Terao's conjecture)

If $\mathcal{A}$ is free then $\mathcal{A}'$ is also free.

In general, for hyperplane arrangements in $\mathbb{A}^n_{\mathbb{K}}$, the module of logarithmic derivations is not completely determined by the combinatorics of the arrangement.

Terao's Conjecture ('80s)

Consider $\mathcal{A}, \mathcal{A}' \subset \mathbb{A}^n_{\mathbb{K}}$ be two hyperplane arrangements containing the origin such that $L(\mathcal{A}) \simeq L(\mathcal{A}')$:

1. In this setting: $\mathcal{D}(\mathcal{A}) = \bigoplus_{d\in\mathbb{N}} \mathcal{D}_d(\mathcal{A})$ by homogeneous components.
2. We said that $\mathcal{A}$ is *free* if $\mathcal{D}(\mathcal{A})$ is a free S-module.

Conjecture (Weak Terao's conjecture)

If $\mathcal{A}$ is free then $\mathcal{A}'$ is also free.

Conjecture (Strong Terao's conjecture)

If $\mathcal{A}$ is free then $\mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}')$ are isomorphic graded modules.

In general, for hyperplane arrangements in $\mathbb{A}^n_{\mathbb{K}}$, the module of logarithmic derivations is not completely determined by the combinatorics of the arrangement.

Terao's Conjecture ('80s)

Consider $\mathcal{A}, \mathcal{A}' \subset \mathbb{A}^n_{\mathbb{K}}$ be two hyperplane arrangements containing the origin such that $L(\mathcal{A}) \simeq L(\mathcal{A}')$:

1. In this setting: $\mathcal{D}(\mathcal{A}) = \bigoplus_{d\in\mathbb{N}} \mathcal{D}_d(\mathcal{A})$ by homogeneous components.
2. We said that $\mathcal{A}$ is *free* if $\mathcal{D}(\mathcal{A})$ is a free S-module.

Conjecture (Weak Terao's conjecture)

If $\mathcal{A}$ is free then $\mathcal{A}'$ is also free.

Conjecture (Strong Terao's conjecture)

If $\mathcal{A}$ is free then $\mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}')$ are isomorphic graded modules.

- What happens for affine line arrangements in the plane ?

Theorem (Seshadri'58)

Any projective module over $\mathbb{K}[x, y]$ *is free.*

- What happens for affine line arrangements in the plane ?

Theorem (Seshadri'58)

Any projective module over $\mathbb{K}[x, y]$ *is free.*

$\Rightarrow \mathcal{D}(\mathcal{A})$ is always free for affine line arrangements!

- What happens for affine line arrangements in the plane ?

Theorem (Seshadri'58)

Any projective module over $\mathbb{K}[x, y]$ *is free.*

$\Rightarrow \mathcal{D}(\mathcal{A})$ is always free for affine line arrangements!

- Two line arrangements $\mathcal{A}, \mathcal{A}'$ with the same combinatorics have isomorphic $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{A}')$ respecting the filtration?

- What happens for affine line arrangements in the plane ?

Theorem (Seshadri'58)

Any projective module over $\mathbb{K}[x, y]$ is free.

$\Rightarrow \mathcal{D}(\mathcal{A})$ is always free for affine line arrangements!

- Two line arrangements $\mathcal{A}, \mathcal{A}'$ with the same combinatorics have isomorphic $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{A}')$ respecting the filtration?

$\rightsquigarrow$ No! We showed that $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$ but $\mathcal{F}_5(\mathcal{Z}_1) \neq \{0\} = \mathcal{F}_5(\mathcal{Z}_2)$.

- What happens for affine line arrangements in the plane ?

Theorem (Seshadri'58)

Any projective module over $\mathbb{K}[x, y]$ is free.

$\Rightarrow \mathcal{D}(\mathcal{A})$ is always free for affine line arrangements!

- Two line arrangements $\mathcal{A}, \mathcal{A}'$ with the same combinatorics have isomorphic $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{A}')$ respecting the filtration?
  - $\rightsquigarrow$ No! We showed that $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$ but $\mathcal{F}_5(\mathcal{Z}_1) \neq \{0\} = \mathcal{F}_5(\mathcal{Z}_2)$.

Perspectives and continuation

- (also with J. Vallés) Use this approach to study the Terao's conjecture for projective line arrangements $\mathbb{P}^2_{\mathbb{K}}$ and central plane arrangements in $\mathbb{A}^3_{\mathbb{K}}$ via projectivization and taking cones from affine line arrangements.
  - Study the meaning of freeness condition in the affine restriction.

Perspectives and continuation

- (also with J. Vallés) Use this approach to study the Terao's conjecture for projective line arrangements $\mathbb{P}^2_{\mathbb{K}}$ and central plane arrangements in $\mathbb{A}^3_{\mathbb{K}}$ via projectivization and taking cones from affine line arrangements.
  - Study the meaning of freeness condition in the affine restriction.
- Investigate $(D(\mathcal{A}), [\cdot, \cdot])$ seen as a Lie sub-algebra of $\mathrm{Der}_{\mathbb{K}}(S)$.

Perspectives and continuation

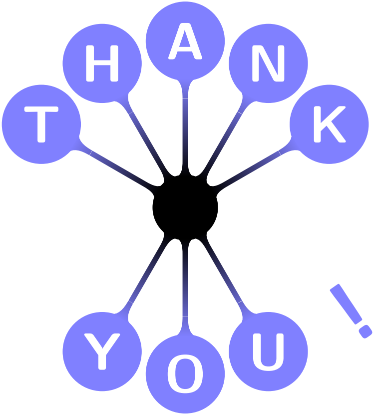
- (also with J. Vallés) Use this approach to study the Terao's conjecture for projective line arrangements $\mathbb{P}^2_{\mathbb{K}}$ and central plane arrangements in $\mathbb{A}^3_{\mathbb{K}}$ via projectivization and taking cones from affine line arrangements.
  - Study the meaning of freeness condition in the affine restriction.
- Investigate $(D(\mathcal{A}), [\cdot,\cdot])$ seen as a Lie sub-algebra of $\mathrm{Der}_{\mathbb{K}}(S)$.
- Developing of a complete package for Sage in order to study the module of logarithmic forms of line arrangements.

Perspectives and continuation

- (also with J. Vallés) Use this approach to study the Terao's conjecture for projective line arrangements $\mathbb{P}^2_{\mathbb{K}}$ and central plane arrangements in $\mathbb{A}^3_{\mathbb{K}}$ via projectivization and taking cones from affine line arrangements.
  - Study the meaning of freeness condition in the affine restriction.
- Investigate $(D(\mathcal{A}), [\cdot,\cdot])$ seen as a Lie sub-algebra of $\mathrm{Der}_{\mathbb{K}}(S)$.
- Developing of a complete package for Sage in order to study the module of logarithmic forms of line arrangements.
- A mixed approach to the Algebraic Hilbert's 16th Problem.

Perspectives and continuation

- (also with J. Vallés) Use this approach to study the Terao's conjecture for projective line arrangements $\mathbb{P}^2_{\mathbb{K}}$ and central plane arrangements in $\mathbb{A}^3_{\mathbb{K}}$ via projectivization and taking cones from affine line arrangements.
  - Study the meaning of freeness condition in the affine restriction.
- Investigate $(D(\mathcal{A}), [\cdot, \cdot])$ seen as a Lie sub-algebra of $\mathrm{Der}_{\mathbb{K}}(S)$.
- Developing of a complete package for `Sage` in order to study the module of logarithmic forms of line arrangements.
- A mixed approach to the Algebraic Hilbert's 16th Problem.


T
H
A
N
K
Y
O
U
!