Algebraic Hilbert's 16th problem and line arrangements

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Introduction

Consider a real planar vector field of degree d given by:

$$\chi = P(x,y)\partial_x + Q(x,y)\partial_y \quad \text{where} \quad P,Q \in \mathbb{R}[x,y] \text{ and } \max\{\deg P, \deg Q\} = d. \tag{1}$$

A real algebraic curve $\mathcal{C} = \{f = 0\}$ with $f \in \mathbb{R}[x,y]$ is called *invariant by* the vector field χ if there exists some $K \in \mathbb{R}[x,y]$ such that

$$\chi f = P(x,y)\frac{\partial f}{\partial x} + Q(x,y)\frac{\partial f}{\partial y} = Kf \tag{2}$$

An *algebraic limit cycle* of χ is a limit cycle, i.e. an isolated periodic orbit of χ, which is contained in an invariant algebraic curve of χ.

Hilbert's 16th problem. The study of limit cycles has been one of the early lines of research in dynamical systems theory starting with Poincaré in 1882. The present statement of the 16th Hilbert problem, restated by S. Smale in his proposal of problems for the 21st century [S]:

Smale's 13th Problem: Is there a bound C on the number of limit cycles of (1) such that $C \leq d^q$ for $q > 0$?

Line arrangements. As in [AGL], a first step in this direction is to study the maximal number of lines for a given polynomial vector field. Our approach is then to focus on the relation between a fixed real line arrangement $\mathcal{A}$ and such a χ fixing $\mathcal{A}$. Thus, we try to understand the influence of the combinatorial structure of an arrangement on the minimal degree of the derivatives of the arrangement (fixing only a finite number of lines). From here, we investigate the bounds of the maximal number of lines invariant by a such χ of fixed degree.

The module of $\mathcal{A}$-derivations

Let $\mathcal{A}$ be a line arrangement with $\mathcal{Q} = \mathcal{Q}(\mathcal{A}) \in \mathbb{R}[x,y]$ its defining polynomial. The central object of our study will be

$$\mathcal{D}(\mathcal{A}) = \{\chi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}[x,y]) \mid \chi\mathcal{Q} \in (\mathcal{Q})\}$$

the *Module of $\mathcal{A}$–derivations*, which is a graded $\mathbb{R}[x,y]$–module: $\mathcal{D}(\mathcal{A}) = \bigoplus_{d\geq 0}\mathcal{D}_d(\mathcal{A})$. Consider the subsets $\mathcal{D}_d^f(\mathcal{A})$ of $\chi \in \mathcal{D}_d(\mathcal{A})$ which **only fix a finite number of lines.**

Problem: Study $d(\mathcal{A}) = \min\{d \in \mathbb{N} \mid \mathcal{D}_d^f(\mathcal{A}) \neq \emptyset\}$.

Our first approach: Combinatorics

We want study the **influence of the combinatorics in** $d(\mathcal{A})$.
First, what about the structure of the objects which we investigate?

Theorem (Structure's Thm)

Let $\mathcal{A}$ an arrangement

- Let $\mathcal{A}$ be a fixed arrangement. The set $\mathcal{D}^d(\mathcal{A})$ forms a $\mathbb{R}$-vectorial subspace of the set of coefficients

$$\mathcal{C} = \left\{(a_{i,j}, b_{i,j}) \in \left(\mathbb{R}^{(d+1)(d+2)/2}\right)^2\right\}.$$

- The set of arrangements of n lines fixed by (1) is an algebraic variety of $\left(\mathbb{P}^2_{\mathbb{R}}\right)^n$.

Denote by $m(\mathcal{A})$ the maximum of the multiplicities of singular points in $\mathcal{A}$. We study the conditions to determine the polynomial vector fields which **only fixes a finite number of lines**:

Theorem

Let $\chi \in \mathcal{D}^d(\mathcal{A})$. If $m(\mathcal{A}) > d+1$ then χ fixes an infinity of lines, i.e. there exist $\mathcal{A}_\infty \supset \mathcal{A}$ such that $|\mathcal{A}_\infty| = \infty$ and $\chi \in \mathcal{D}^d(\mathcal{A}_\infty)$.

Theorem

A vector field (1) fixes an infinity of lines if and only if we are in one of the following cases:

1. χ is orthogonal to the radial vector field center in a point (x_0, y_0), i.e. $(y-y_0)P(x,y) - (x-x_0)Q(x,y) = 0$.
2. There is $\lambda, \nu \in \mathbb{R}$ such that $\nu P(x,y) = \lambda Q(x,y)$.

These two theorems allows us to reduce our study of the degree to few polynomial vector fields.

Perspectives

We are interested in:

1. Give a formula or a bound for $d(\mathcal{A})$ in some cases.
2. Determine if $d(\mathcal{A})$ depends only of the combinatorics of $\mathcal{A}$.
3. A future generalization to arrangements of smooth curves.

References

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