A semi-canonical reduction for periods of Kontsevich-Zagier

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Contenidos

Part I

Introduction

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represented taking $\mathbb{Q}$–basis by the *period matrix* $\Pi = \left(\int_{\gamma_i} \omega_j\right)_{i,j=1,\ldots,s}$.

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- QUESTION: Could the comparison isomorphism be induced by an isomorphism $H^\bullet_{\mathrm{dR}}(X, Y; \mathbb{Q}) \xrightarrow{\simeq} H^\bullet_{\mathrm{B}}(X, Y; \mathbb{Q})$?
  - NO! If $X = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$, $Y = \emptyset$ and $\gamma = S^1 \subset \mathbb{C}^*$:

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Periods of Kontsevich-Zagier (2001)

Let $\mathbb{R}_{\text{alg}}$ be the field of algebraic numbers.

A set $S \subset \mathbb{R}^d$ is called $\mathbb{R}_{\text{alg}}$–*semi-algebraic* if can be described as finite unions of sets $\{f_1 *_1 0, \ldots f_s *_s 0\}$, where $f_i \in \mathbb{R}_{\text{alg}}[x_1, \ldots, x_d]$ and $*_i \in \{=, >\}$ for $i = 1, \ldots, s$.

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A *period of Kontsevich-Zagier* (or *effective period*) is a complex number whose real and imaginary parts are values of absolutely convergent integrals of the form

$$\mathcal{I}(S, P/Q) = \int_S \frac{P(x_1, \ldots, x_d)}{Q(x_1, \ldots, x_d)} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$$

where $S \subset \mathbb{R}^d$ is a d–dimensional $\mathbb{R}_{\text{alg}}$–semi-algebraic set and $P/Q \in \mathbb{R}_{\text{alg}}(x_1, \ldots, x_d)$.

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Examples of numbers in $\mathcal{P}_{\text{KZ}}$

1. Algebraic numbers: $\alpha = \int_0^\alpha \mathrm{d}x$, $\forall \alpha \in \mathbb{R}_{\mathsf{alg}}$.
2. As a first transcendental number

$$\pi = \int_{\{x^2+y^2\leq 1\}} 1 \, \mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, \mathrm{d}x = \int_{\{(1-x^2)y^2<1\}} \frac{\mathrm{d}x\mathrm{d}y}{2}$$

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Open problems and conjectures

From the foundational paper:

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Conjecture (Konsevich-Zagier periods conjecture)

If a real period admits two integral representations, then we can pass from one formulation to the other using only three operations (called the KZ–rules):

- *integral additions by domains or integrands.*
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- *Stokes formula.*

Moreover, these operations should respect the class of the objects previously defined.

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PART II

A SEMI-CANONICAL REDUCTION FOR PERIODS

"Periods of Kontsevich-Zagier I: A semi-canonical reduction.",
arXiv:1509.01097, 26 pags., (Preprint)

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Our principal result:

Theorem (Semi-canonical reduction)

Let $p \in \mathcal{P}_{\mathrm{KZ}}$ be non-zero given in an integral form $\mathcal{I}(S, P/Q)$ in $\mathbb{R}^d$. Then there exists an effective algorithm satisfying the KZ–rules such that $\mathcal{I}(S, P/Q)$ can be written as

$$p = \operatorname{sgn}(p) \cdot \operatorname{vol}_{d+1}(K),$$

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Compactification

We define the *projective closure* of a semi-algebraic set $S \subset \mathbb{R}^d$ by the topological closure of the inclusion of $S \hookrightarrow \mathbb{P}^d_{\mathbb{R}}$.

Theorem

$\mathbb{P}^d_{\mathbb{R}}$ can be constructed as the gluing of $C_1, \ldots, C_{d+1}$ affine unit hypercubes through their opposite faces, and such that the Zariski closure of $\bigcup_{i,j=0}^{d}(C_i \cap C_j)$ is the hyperplane arrangement

$$\mathcal{A} = \{x_i^2 - x_j^2 = 0 \mid 0 \leq i < j \leq d\} \subset \mathbb{P}^d_{\mathbb{R}}$$

$\rightsquigarrow D = D_1 \sqcup \ldots \sqcup D_{d+1}$ affine compact up to $(d-1)$–dim semi-algebraic sets.

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Resolution of poles

We can assume that we are dealing with integrals $\mathcal{I}(S, P/Q)$ with compact domains.

Let W_0 be a smooth real algebraic variety defined over $\mathbb{R}_{\text{alg}}$. Let $S \subset W_0$ be a compact semi-algebraic set in W_0 and ω a top differential rational form in W_0. Denote by $\partial_z S$ the Zariski closure of ∂S and by $Z(\omega)$ and $P(\omega)$ the real zero and pole locus of ω, respectively.

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We use embedded resolution of singularities to send the poles "far away" from ∂S.

Proposition (Geometric criterion for convergence)

The integral $\int_S \omega$ converges absolutely if and only if there exist a finite sequence of blow-ups $\pi = \pi_r \circ \cdots \circ \pi_1 : W_r \to W_0$ over smooth centers such that $\widetilde{S} \cap P(\pi^* \omega) = \emptyset$, where $\widetilde{S}$ the strict transform of S.

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Resolution of poles

We can assume that we are dealing with integrals $\mathcal{I}(S, P/Q)$ with compact domains.
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Compact domains in $\mathbb{R}^2$ and tangent cones

This case is more easy to manipulate:

- Blow-ups over points $p \in \partial S$.
- The compacity of the domain can be controlled *a priori* using the *tangent cone* $T_p(\partial_z S)$ *at* p of $\partial_z S$.

Proposition

Let $p \in \partial S$ and suppose that there exists a line L such that $\overline{S} \cap L = \{p\}$. If $L \notin T_p(\partial_z S)$ then there exist a Zariski open $U \subset \widehat{\mathbb{R}}^2$ such that $\widetilde{S}^{\tau} \cap U$ is compact.

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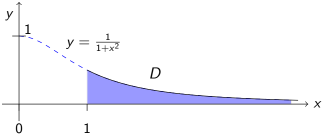

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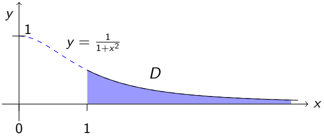

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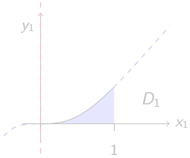

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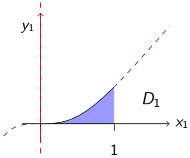

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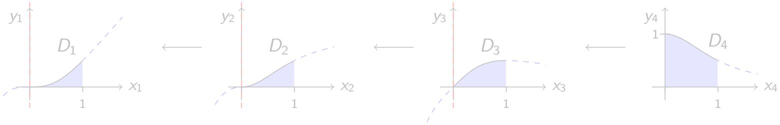

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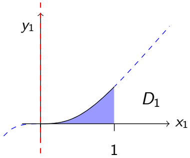

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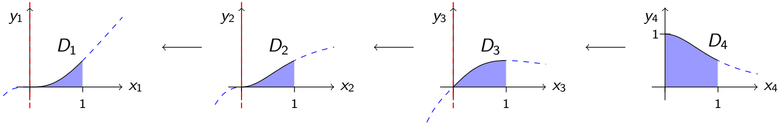

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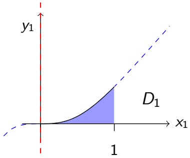

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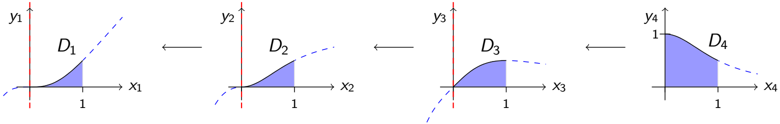

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Part III

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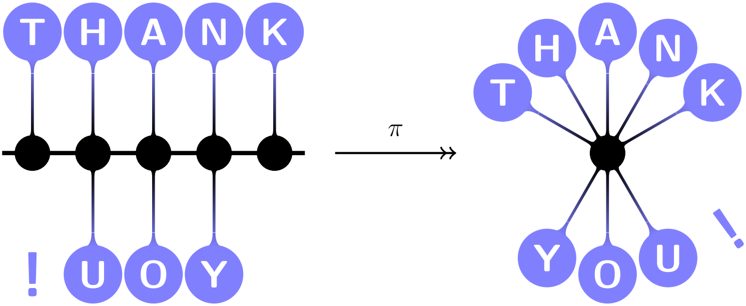
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